riant subg...

§ 4 **Quotient Groups.** We are now going to introduce the very important concept of *Quotient Groups*.

Let H be any normal subgroup of a group G. If $a \in G$, then Ha is a right coset of H in G. Also H being normal in G, the left coset aH will be equal to the right coset Ha. Thus there is no distinction between right and left cosets. So we can say that Ha is a coset of H in G. Let G/H be the collection of all cosets of H in G i.e., let
$$G/H = \{Ha : a \in G\}.$$

If $a, b \in G$, then we have

$$(Ha)(Hb) = H(aH)b$$
$$= H(Ha)b \quad [\because Ha = aH \text{ as } H \text{ is normal}]$$
$$= HHab = Hab. \quad [\because HH = H]$$

Now $ab \in G$, therefore the product of two cosets of H in G is again a coset of H in G. We shall presently see that the set G/H is a group with respect to multiplication of cosets.

Theorem. *The set of all cosets of a normal subgroup is a group with respect to multiplication of complexes as the composition.*

(Meerut 1979; Kanpur 87; Nagarjuna 80; Patna 87)

Proof. Let H be a normal subgroup of a group G. Since H is normal in G, therefore each right coset will be equal to the corresponding left coset. Thus there is no distinction between right and left cosets and we shall call them simply as cosets. Let G/H be the collection of all cosets of H in G i.e., let
$$G/H = \{Ha : a \in G\}.$$

Closure Property. Let $a, b \in G$. Then
$$(Ha)(Hb) = H(aH)b = H(Ha)b = HHab = Hab.$$

Since $ab \in G$, therefore Hab is also a coset of H in G. So $Hab \in G/H$. Thus G/H is closed with respect to coset multiplication.

Associativity. Let $a, b, c \in G$. Then $Ha, Hb, Hc \in G/H$. We have $Ha[(Hb)(Hc)] = Ha(Hbc) = Ha(bc)$

$=H(ab)c$ $[\because a(bc)=(ab)c]$

$=(Hab)Hc=[(Ha)(Hb)]Hc.$

Thus the product in G/H satisfies the associative law.

Existence of Identity. We have $H=He\in G/H$. Also if Ha is any element of G/H, then

$$H(Ha)=(He)(Ha)=Hea=Ha \text{ and similarly}$$

$$(Ha)H=(Ha)(He)=Hae=Ha.$$

Therefore the coset H is the identity element.

Existence of Inverse. Let $Ha\in G/H$. Then $Ha^{-1}\in G/H$.

We have $(Ha)(Ha^{-1})=Haa^{-1}=He=H$

and $(Ha^{-1})(Ha)=Ha^{-1}a=He=H.$

$\therefore$ The coset Ha^{-1} is the inverse of Ha *i.e.*, $(Ha)^{-1}=Ha^{-1}$.

Thus each element of G/H possesses inverse.

Hence G/H is a group with respect to product of cosets.

Definition. Quotient Group.

(I.A.S. 1970; Meerut 79; Garhwal 76; Mysore 70)

If G is a group and H is a normal subgroup of G, then the set G/H of all cosets of H in G is a group with respect to multiplication of cosets. It is called the **quotient group or factor group** *of G by H.*

The identity element of the quotient group G/H is H.

Solved Examples

Ex. 1. *Let* **I** *be the additive group of integers. Let H be a subgroup of* **I** *such that $H=\{mx : x\in \mathbf{I}\}$ where m is a fixed positive integer. Write the elements of the quotient group $\mathbf{I}/H$. Also prepare a composition table for $\mathbf{I}/H$ when $m=5$.* **(Meerut 1976)**

Solution. Since **I** is an abelian group, therefore H is normal in **I**. The elements of $\mathbf{I}/H$ are the cosets of H in **I** namely

$$H+0=H=\{\ldots -2m, -m, 0, m, 2m, \ldots\}$$

$$H+1=\{\ldots, -2m+1, -m+1, 1, m+1, 2m+1, \ldots\}$$

$$H+2=\{\ldots, -2m+2, -m+2, 2, m+2, 2m+2, \ldots\}$$

... ...

... ...

$$H+(m-2)=\{\ldots, -m-2, -2, m-2, 2m-2, 3m-2, \ldots\}$$

$$H+(m-1)=\{\ldots, -m-1, -1, m-1, 2m-1, 3m-1, \ldots\}.$$

These are the only distinct cosets of H in **I**. Because if s is any integer, then by division algorithm there exist integers q and r such that $s=mq+r$ where $0\leq r\leq m-1$.

We have $H+s=H+mq+r$

$=H+r$ $[\because mq\in H$ and this gives $H+mq=H]$.

Thus $H+s$ is one of the above m cosets of H in **I**. Thus there are m distinct elements in the set **I**$/H$.

When $m=5$, the distinct elements in **I**$/H$ are

$$H,\ H+1,\ H+2,\ H+3,\ H+4.$$

If $a, b \in$ **I**, then $(H+a)+(H+b)=H+(a+b)$.

Also $H+a=H+b \Leftrightarrow a-b \in H$. Thus $H+2=H+7$, $H+3=H+8$ and so on.

Hence the composition table for **I**$/H$ is as given below :

	H	$H+1$	$H+2$	$H+3$	$H+4$
H	H	$H+1$	$H+2$	$H+3$	$H+4$
$H+1$	$H+1$	$H+2$	$H+3$	$H+4$	H
$H+2$	$H+2$	$H+3$	$H+4$	H	$H+1$
$H+3$	$H+3$	$H+4$	H	$H+1$	$H+2$
$H+4$	$H+4$	H	$H+1$	$H+2$	$H+3$

Ex. 2. *Let P_3 be the symmetric group on three symbols a, b, c and A_3 be the alternating group on three symbols a, b, c. Form the composition table for the quotient group P_3/A_3.* **(Rajasthan 1977)**

Solution. Let $P_3=\{f_1, f_2, f_3, f_4, f_5, f_6\}$ where $f_1=$identity permutation, $f_2=(ab)$, $f_3=(bc)$, $f_4=(ca)$, $f_5=(abc)$, $f_6=(acb)$.

We have $A_3=$set of even permutations belonging to P_3. Thus $A_3=\{f_1, f_5, f_6\}$.

A_3 is a normal subgroup of P_3 as we have already proved.

The elements of P_3/A_3 are the cosets of A_3 in P_3. By Lagrange's theorem A_3 will have only two distinct cosets in P_3. One is A_3 itself and the other is $A_3 f_2$. Thus A_3, $A_3 f_2$ are the only two elements of P_3/A_3. It should be noted that

$$A_3 f_5=A_3 f_6=A_3,\ A_3 f_2=A_3 f_3=A_3 f_4.$$

The composition table for P_3/A_3 is as given below :

	A_3	$A_3 f_2$
A_3	A_3	$A_3 f_2$
$A_3 f_2$	$A_3 f_2$	A_3

[Note that $(A_3 f_2)(A_3 f_2)$
$=A_3 f_2 f_2=A_3 f_1=A_3$]

Ex. 3. *If G is a finite group and H is a normal subgroup of G, then* $o(G/H)=\frac{o(G)}{o(H)}$. **(Meerut 1980, 82, 90; Kumayun 77)**

Solution. We have

$$o(G/H)=\text{number of distinct right cosets of } H \text{ in } G$$

$$=\frac{\text{Number of elements in } G}{\text{Number of elements in } H} \quad \text{[by Lagrange's theorem]}$$

$$=\frac{o(G)}{o(H)}.$$

Ex. 4. *Show that every quotient group of an abelian group is abelian and the converse is not true.* **(Rajasthan 1974)**

Solution. Let G be an abelian group and H be a subgroup of G. Then H is a normal subgroup of G. If $a, b \in G$, then Ha, Hb are any two elements of G/H. We have

$$(Ha)(Hb)=Hab=Hba \quad [\because G \text{ is abelian} \Rightarrow ab=ba]$$

$$=(Hb)(Ha).$$

$\therefore$ G/H is abelian.

The converse is not true. For example if P_3 be the symmetric group of degree 3 and A_3 be the alternating group of degree 3, then P_3/A_3 is an abelian group while P_3 is not an abelian group. The group P_3/A_3 is of order 2, and every group of order 2 is abelian.

Ex. 5. *If N is normal in G and* $a \in G$ *is of order n, prove that the order, m, of Na in G/N is a divisor of n.*

Solution. The identity of the quotient group G/N is N. If e is the identity of G, then $o(a)=n \Rightarrow a^n=e$.

$$\therefore \quad Na^n=Ne=N.$$

But $Na^n=N(aaa\ldots\text{upto } n \text{ times})$

$$=(Na)(Na)(Na)\ldots\text{upto } n \text{ times}=(Na)^n.$$

Thus $(Na)^n=N$ (identity of G/N).

Since order of Na in G/N is m and $(Na)^n=N$, identity of G/N, therefore m must be a divisor of n. [Refer theorem 4, page 115]

Ex. 6. *Show that every quotient group of a cyclic group is cyclic and the converse is not true.* **(Jodhpur 1970; Meerut 78)**

Solution. Let G be a cyclic group and a be a generator of G. Let H be a subgroup of G. Since every cyclic group is abelian, therefore H is a normal subgroup of G. Let a^n be any element of

G where n is some integer. Then Ha^n is any element of G/H. As can be easily seen $Ha^n = (Ha)^n$ for every integer n. Therefore G/H is a cyclic group and Ha is a generator of it.

The converse is not true. For example P_3/A_3 is cyclic while P_3 is not cyclic.

Ex. 7. *Let Z be the centre of a group G. If $a \in Z$, then prove that the cyclic subgroup $\{a\}$ of G generated by a is a normal subgroup of G.*

Solution. We have

$Z = \{z \in G : zx = xz \ \forall \ x \in G\}$. Let $a \in Z$ and let $H = \{a\}$ be the cyclic subgroup of G generated by a. Let h be any element of H. Then $h = a^n$ for some integer n.

Let x be any element of G. We have

$$xhx^{-1} = xa^n x^{-1}$$
$$= (xax^{-1})^n \qquad \text{[See Ex. 3 page 117 of first chapter on groups]}$$
$$= (axx^{-1})^n \qquad [\because a \in Z \Rightarrow ax = xa]$$
$$= (ae)^n = a^n \in H.$$

Thus $xhx^{-1} \in H \ \forall \ h \in H$ and $\forall \ x \in G$.

$\therefore$ H is a normal subgroup of G.

Ex. 8. *Let a be any element of G. Show that the cyclic subgroup of G generated by a is a normal subgroup of the normalizer of a.* **(Punjab 1970)**

Solution. We have the normalizer of a

$$= N(a) = \{x \in G : xa = ax\}.$$

Let H be the cyclic subgroup of G generated by a. Let h be any element of H. Then $h = a^n$ where n is some integer. We have

$$a^n a = a^{n+1} = aa^n.$$

$\therefore$ $a^n = h \in N(a)$.

Now $N(a)$ and H are subgroups of G. Also $h \in H \Rightarrow h \in N(a)$. Therefore $H \subseteq N(a)$. Hence H is a subgroup of $N(a)$.

Now to prove that H is a normal subgroup of $N(a)$. Let x be any element of $N(a)$ and $h = a^n$ be any element of H. We have

$$xhx^{-1} = xa^n x^{-1} = (xax^{-1})^n$$
$$= (axx^{-1})^n \qquad [\because x \in N(a) \Rightarrow ax = xa]$$
$$= (ae)^n = a^n \in H.$$

$\therefore$ H is a normal subgroup of $N(a)$.

Ex. 9. *Show that two elements are conjugate if and only if they can be put in the form xy and yx respectively where x and y are suitable elements of G.* **(Punjab 1970)**

Solution. Let a, b be two conjugate elements of a group G.

Then $a=c^{-1}bc$ for some $c \in G$.

Let $c^{-1}b=x$ and $c=y$. Then $a=xy$. Also

$$yx=c\,(c^{-1}b)=(cc^{-1})\,b=eb=b.$$

Conversely suppose that $a=xy$ and $b=yx$. We have

$$b=yx \Rightarrow y^{-1}b=y^{-1}yx \Rightarrow y^{-1}b=x.$$

Now $a=xy$

$$\Rightarrow a=y^{-1}\,by \Rightarrow a \text{ and } b \text{ are conjugate elements.}$$

Ex. 10. *Give an example to show that in a group G the normaliser of an element is not necessarily a normal subgroup of G.*

(Meerut 1985, 91; B.H.U. 88)

Solution. Consider the group S_3, the symmetric group of permutations on three symbols a, b, c. We have $S_3=\{I, (ab), (bc), (ca), (abc), (acb)\}$. Let $N\,(ab)$ denote the normaliser of the element $(ab) \in S_3$. We shall show that $N\,(ab)$ is not a normal subgroup of S_3. Let us calculate the elements of $N(ab)$. Obviously $(ab) \in N\,(ab)$. Also $I \in N\,(ab)$ because $I\,(ab)=(ab)\,I$

Now $(bc)\,(ab)=(abc)$ and $(ab)\,(bc)=(acb)$. Thus (bc) does not commute with (ab). Therefore $(bc) \notin N\,(ab)$. Again

$$(ca)\,(ab)=(acb) \text{ and } (ab)\,(ca)=(abc).$$

Thus $(ca)\,(ab) \neq (ab)\,(ca)$ and therefore $(ca) \notin N(ab)$. Similarly we can verify that $(abc) \notin N\,(ab)$ and $(acb) \notin N\,(ab)$. Hence

$$N\,(ab)=\{I, (ab)\}.$$

Now we shall show that $N\,(ab)$ is not a normal subgroup of S_3. Take the element $(bc) \in S_3$ and the element $(ab) \in N\,(ab)$. We have $(bc)\,(ab)\,(bc)^{-1}=(bc)\,(ab)\,(cb)=(abc)\,(cb)=(ac) \notin N\,(ab)$. Therefore $N(ab)$ is not a normal subgroup of S_3.

Ex. 11. *Let N_1 and N_2 be two normal subgroups of a group G. Prove that $G/N_1=G/N_2$ if and only if $N_1=N_2$.*

Solution. If $N_1=N_2$, then obviously $G/N_1=G/N_2$.

Conversely suppose that $G/N_1=G/N_2$. Then we are to prove that $N_1=N_2$. We have $N_1 \in G/N_1$. But $G/N_1=G/N_2$. Therefore $N_1 \in G/N_2$ *i.e.*, N_1 is equal to some coset of N_2 in G. But two cosets of N_2 in G are either disjoint or identical. Since $e \in N_1$ and $e \in N_2$, therefore N_1 and N_2 are not disjoint. So we must have

$$N_1=N_2.$$

Ex. 12. *Let Z denote the centre of a group G. If G/Z is cyclic prove that G is abelian.* **(Meerut 1978, 81; I.C S. 90; Guru Nanak 89, Madurai 88)**

Solution It is given that G/Z is cyclic. Let Zg be a generator of the cyclic group G/Z where g is some element of G.

Let $a, b \in G$. Then to prove that $ab=ba$. Since $a \in G$, therefore $Za \in G/Z$. But G/Z is cyclic having Zg as a generator. Therefore there exists some integer m such that $Za=(Zg)^m=Zg^m$, because Z is a normal subgroup of G. Now $a \in Za$. Therefore

$$Za=Zg^m \Rightarrow a \in Zg^m \Rightarrow a=z_1 g^m \text{ for some } z_1 \in Z.$$

Similarly $b=z_2 g^n$ where $z_2 \in Z$ and n is some integer.

Now $ab=(z_1 g^m)(z_2 g^n)=z_1 g^m z_2 g^n$

$=z_1 z_2 g^m g^n$ $[\because z_2 \in Z \Rightarrow z_2 g^m = g^m z_2]$

$=z_1 z_2 g^{m+n}$.

Again $ba=z_2 g^n z_1 g^m=z_2 z_1 g^n g^m=z_2 z_1 g^{n+m}$

$=z_1 z_2 g^{m+n}$ $[\because z_1 \in Z \Rightarrow z_1 z_2 = z_2 z_1]$

$\therefore ab=ba$.

Since $ab=ba \ \forall \ a, b \in G$, therefore G is abelian.

Ex. 13 *If p is a prime number and G is a non-abelian group of order p^3, show that the centre of G has exactly p elements.* **(Madras 1983)**

Solution. Let Z denote the centre of G. Since $o(G)=p^3$ where p is a prime number, therefore $Z \neq \{e\}$ *i.e.*, $o(Z)>1$. But Z is a subgroup of G, therefore $o(Z)$ must be a divisor of $o(G)$ *i.e.*, $o(Z)$ must be a divisor of p^3. Since p is prime, therefore either $o(Z)=p$ or p^2 or p^3.

If $o(Z)=p^3=o(G)$, then $Z=G$ and so G is abelian which contradicts the hypothesis that G is non-abelian. So $o(Z)$ cannot be p^3.

If $o(Z)=p^2$, then $o(G/Z)=o(G)/o(Z)=p^3/p^2=p$ *i.e.*, G/Z is a group of prime order p and so is cyclic. But if G/Z is cyclic, then G is abelian which again contradicts the hypothesis. So $o(Z)$ cannot be p^2.

Hence the only possibility is that $o(Z)=p$ *i.e.*, the centre of G has exactly p elements.